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1988 J. Phys. A: Math. Gen. 21 L995

(http://iopscience.iop.org/0305-4470/21/20/007)

J. Phys. A: Math. Gen. 21 (1988) L995–L1001. Printed in the UK

LETTER TO THE EDITOR

Distribution of growth probabilities in fluid flow and diffusion-limited aggregation

P Ramanlal and L M Sander

Physics Department, University of Michigan, Ann Arbor, MI 48109-1120, USA

Received 10 May 1988

Abstract. We examine the role of extrinsic noise in diffusion-limited aggregation (DLA) and a deterministic continuum theory derived from the work of Paterson on fluid flow in a Hele-Shaw cell. We show the distribution of growth probabilities for the deterministic model to be essentially the same as DLA and predict that the asymptotic value of the fractal dimension for off-lattice DLA may be closer to $D \sim 1.65 \pm 0.03$, as opposed to $D = 1.71$. Our results confirm that extrinsic noise is not essential to create the characteristic properties of DLA in models that are variations of DLA.

When a non-equilibrium object grows by aggregation, scaling symmetry sometimes seems to result [1]. There are several such processes which have been idealised as computer simulation models. The best known of these is diffusion-limited aggregation (DLA) [2, 3]. It is of great interest to see whether this simple computer model has a counterpart in the real world. For example, electro-deposition of metals of an electrode can be thought of as DLA [4–6]. In this case the visual impression given by the objects formed, the fractal dimension and a microscopic description of the deposition process [4] strongly resemble DLA.

Another process which has been studied a good deal is unstable flow in a porous medium or a Hele–Shaw cell [7–9]. Here too DLA dynamics has been invoked. This derives from the work of Paterson [10] who pointed out that, because the viscous fluid moves satisfying Darcy's law, the equations satisfied (see below) are those of a continuum model of DLA. In fact, in some cases [8] (non-Newtonian flow of slowly mixing fluids) the visual similarity to computer simulations is striking. Probably, in this case, experimental noise is amplified by shear thinning. However, for Newtonian immiscible fluids, the situation is not so clear. The patterns may scale like DLA for very large sizes, but neither experiments nor computer solutions of the fluid flow equations [11] achieve the necessary size. This problem is of interest because, if a deterministic continuum theory scales like DLA, then one is forced to address the issue of the source of fluctuations needed to drive the system. In numerical simulations of DLA, the source is generally believed to be the persistant noise associated with the random arrival of particles. In this letter we consider a deterministic process generating DLA-like objects in the absence of extrinsic noise—a process which generates its own noise due to sensitivity to initial conditions, in much the same way as chaotic dynamical systems.

The present authors [11] produced such a process which is closely related to Hele–Shaw flow with altered boundary conditions. We found a fractal dimension near that of DLA, of $D \sim 1.73$, and a ramified pattern (see figure 1(a)). However, the fractal

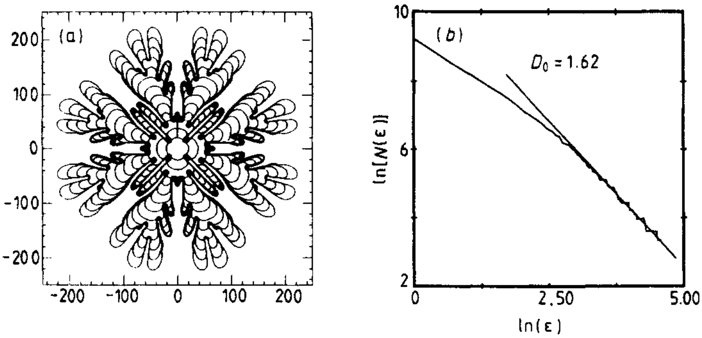

Figure 1. (*a*) Solution of the deterministic growth equations. (*b*) Determination of the fractal dimension of the interface.

dimension is a very incomplete measure of the growth. A more complete description is given by the distribution of the growth probabilities on the surface [12, 13]; these form a fractal measure. In this letter we will demonstrate that, in addition to the similarity in the value of *D*, the deterministic model of [11] has essentially the same fractal measure as off-lattice DLA†. This, we believe, adds considerably to our understanding of the role of extrinsic noise in DLA-like processes, even though it dominates in practical situations [8]. Our study is similar to that of Kantor *et al* [15] in that the quantitative nature of the external source of fluctuations is relatively unimportant in DLA-like processes. In fact, in our model there is no external source. Another surprising result is that the multifractal analysis of off-lattice DLA and the deterministic model suggests an asymptotic value of $\sim 1.65 \pm 0.03$ for the fractal dimension.

We state here the main results of the formalism to evaluate the fractal measure of the different growth probabilities on the interface. In general, one can construct an infinite hierarchy of generalised dimensions [12, 13]

$$D_q = \lim_{\varepsilon \to 0} \frac{1}{(q-1)} \log\left(\sum_{i=1}^{N(\varepsilon)} p_i^q(\varepsilon)\right) (\log \varepsilon)^{-1} \qquad (1)$$

where $N(\varepsilon)$ is the number of coverings of size ε needed to cover the boundary. For the fluid flow problem $\{p_i(\varepsilon)\}$ are the normalised interface velocities. Assuming the probability $p_i(\varepsilon)$ can be expressed as a power-law singularity of the form $p_i(\varepsilon) \sim \varepsilon^{\alpha-1}$ and that $f(\alpha)$ is the fractal dimension of the set of singularities of strength α, we have

$$D_q = \frac{1}{(q-1)} [q\alpha(q) - f(\alpha(q))] \qquad (2a)$$

$$\alpha(q) = \frac{\mathrm{d}}{\mathrm{d}_q} (q-1) D_q. \qquad (2b)$$

Evaluating D_q numerically, the functions $\alpha(q)$, $f(q)$ and $f(\alpha)$ may be subsequently obtained using (2). The asymptotic value for the fractal dimension can now be written down: $D = 1 + \alpha_{\min}(q \to \infty)$. The fractal dimension of the interface itself is $D_0 = f_{\max}(q \to 0)$.

† A similar experimental study was conducted in [14]. However, this does not bear on the main point of this letter—the role of noise.

To apply the above formalism to the case of viscous fingering in a radial Hele-Shaw cell, we solve the following equations for the average growth [11]:

$$\nabla^2 u = 0 \tag{3a}$$

$$v_n = -\hat{n} \cdot \nabla u / 4\pi|_s \tag{3b}$$

$$u(R_0) = 0 \tag{3c}$$

$$u(x_s) = 1 - \kappa^N(x_s) \tag{3d}$$

$$u(x_{\mathrm{int}}) = 1 \tag{3e}$$

to obtain the interface velocity v_n normalised to give the growth probabilities $\{p_i(\varepsilon)\}$ and the interface position $\{r_i(\varepsilon)\}$. The case $N = 1$ corresponds to the radial fluid flow problem and the case $N \to \infty$ to the zero surface tension limit of DLA with a lower length cutoff of unity. The difficulty in obtaining a multi-branched structure characteristic of fluid flow patterns using (3) with $N = 1$ is strictly computational. This was overcome by modifying the equations such that N is some positive odd integer greater than unity. The result was twofold. Firstly, this introduces a non-trivial scaling of all lengths from $R \to R^N$, making it possible to generate larger-scale numerical solutions to (3), reminiscent of actual fluid flow patterns. Secondly, the scheme allows for a continuous approach to the zero surface tension limit of DLA, a limit that has defied numerical progress in continuum models for DLA. Figure 1(a) is an example of a solution to (3) with $N = 5$. The symmetry of the cluster generated is imposed to facilitate the numerics and is due to the symmetric initial condition and the deterministic algorithm. Since the above formalism may be applied effectively only in cases where the interface exhibits fractal behaviour over a reasonable range of length scales, it cannot be applied effectively to numerical solutions for $N = 1$, since the fractal window in this case is prohibitively small. In fact, figure 1(a) represents the computational limit, even in the case $N = 5$.

The solution to (3) yields the quantities $\{p_i(\varepsilon)\}$ and $\{r_i(\varepsilon)\}$, sufficient to evaluate D_q. In figure 1(b) we plot the number of coverings $N(\varepsilon)$ of size ε needed to cover the boundary. The fractal dimension D_0 of the curve, given by $N(\varepsilon) \sim \varepsilon^{-D_0}$, is estimated to be $D_0 \sim 1.627$ over a range of $16 < \varepsilon < 100$. For $\varepsilon < 16$ the interface exhibits a one-dimensional behaviour, as it must. Due to computational constraints, we are unable to obtain a fractal window as large as in numerical simulations of DLA.

The function D_q was evaluated numerically for the final stage of growth in figure 1(a), essentially by plotting $\log(\Sigma_{i=1}^{N(\varepsilon)} p_i^q(\varepsilon))$ against $\log(\varepsilon)$ and reading off the slope in the region $16 < \varepsilon < 100$. The functions $\alpha(q)$, $f(q)$ and $f(\alpha)$ are subsequently evaluated using (2). The results are plotted in figure 2 and are found to have the generally predicted characteristics [13]. Note that the point $q = 1$ is numerically singular. Data were also obtained for intermediate stages of growth, and the relevant information is given in table 1. Note that $f_{\max}$ is the fractal dimension, D_0, of the interface and is seen to evolve from its initial value of unity to a final value of ~ 1.627. On the other hand, $\alpha_{\min}$ predicts the asymptotic value for the fractal dimension of the bulk of the structure generated: $D = 1 + \alpha_{\min}$, seen to decrease from $D \sim 2.0$ to $D \sim 1.684$. If we assume the structure generated in figure 1(a) to be scale invariant in the long-time limit, with a single scaling exponent, then we expect to find $D = D_0$, or equivalently $f_{\max} = 1 + \alpha_{\min}$, as is the case for DLA clusters where there is no distinction between the interface and the interior region. For the deterministic model our prediction for the asymptotic value of the fractal dimension is in the range $1.627 < D = 1 + \alpha_{\min} < 1.684$, even though the fractal dimension at the present stage of growth is ~ 1.73.

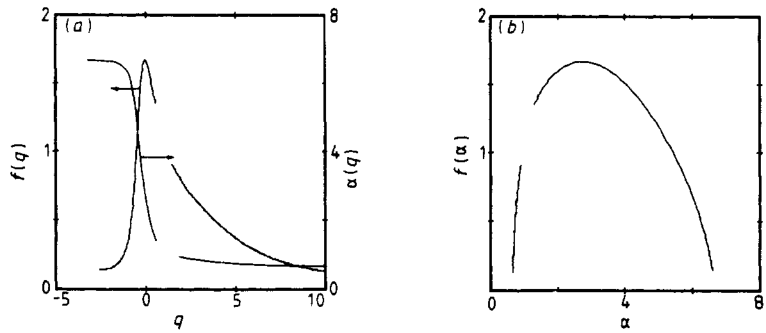

Figure 2. (a) $\alpha(q)$ and $f(q)$ for the deterministic model. (b) $f(\alpha)$ for the deterministic model.

Table 1. The fractal dimension of the interface ($f_{\max}$) and the strongest singularity ($\alpha_{\min}$) for intermediate stages of growth.

$f_{\max}$	$\alpha_{\min}$
1.054	0.949
1.261	0.872
1.371	0.814
1.527	0.742
1.627	0.684

We now examine the fractal measure of the growth probabilities of DLA clusters. This has been investigated by Halsey *et al* [16] where the function D_q was evaluated numerically for $2 \leq q \leq 8$ and the estimate $D = 1.71$ obtained, consistent with numerical simulation results. In addition to D_q, a second family of exponents γ_j, defined by $L^{D_q} \sim M^{\gamma_j}$, have been evaluated for DLA clusters [18]. L is the characteristic size of clusters, M is the cluster mass and $j = q - 1$. We reproduce these results in the first two columns of table 2. In addition, with $M \sim L^D$ we have $D_q = D\gamma_j$. In order to verify this relation we assume $D = 1.71$, consistent with present estimates. The results are expressed in column 3 of table 2.

Table 2. Results for D_q from numerical simulations of DLA and the deterministic growth model.

q	D_q DLA	γ_j DLA	$D\gamma_j$ $D = 1.71$	$D\gamma_j$ $D = 1.63$	D_q deterministic model
2	0.980 ± 0.010	0.529 ± 0.007	0.905	0.873	0.923 ± 0.034
3	0.856 ± 0.007	0.500 ± 0.007	0.855	0.825	0.867 ± 0.034
4	0.810 ± 0.006	0.481 ± 0.007	0.823	0.794	0.832 ± 0.034
5	0.782 ± 0.006	0.468 ± 0.007	0.800	0.772	0.806 ± 0.035
6	0.763 ± 0.008	0.458 ± 0.007	0.783	0.756	0.787 ± 0.035
7	0.748 ± 0.010	0.450 ± 0.007	0.770	0.743	0.772 ± 0.035
8	0.735 ± 0.010	0.444 ± 0.007	0.760	0.733	0.761 ± 0.035
9		0.439 ± 0.007	0.751	0.724	0.751 ± 0.035
∞		0.396 ± 0.009	0.677	0.653	0.684

In figure 3 the full curves 1 and 2 represent D_q and $D\gamma_j$, respectively, for DLA clusters, using $D=1.71$. The question is: do they verify the relation $D_q = D\gamma_j$? This is unclear in the region of available data $2 \leqslant q \leqslant 8$. This is not surprising in view of the fact, as pointed out by Meaking *et al* [18], that the numerical estimates for the harmonic measures yield poor results for small $q(j)$ and are most accurate for large $q(j)$ because of the difficulty in determining the measure in the strongly screened interior regions of clusters. As such, we examine D_q for large q. This is accomplished by determining the limiting behaviour of the curves in figure 3. Using (2) and the extremal condition $\mathrm{d}(q\alpha' - f(\alpha'))/\mathrm{d}\alpha'|_{\alpha'=\alpha(q)} = 0$ we get

$$m = \frac{\mathrm{d}(\log D_q)}{\mathrm{d}\{\log[q/(q-1)]\}} = 1 - \frac{f(q)}{D_q}. \tag{4}$$

The general behaviour of D_q, m, and $\mathrm{d}m/\mathrm{d}\{\log[q(q-1)]\}$ is summarised in the following.

(*a*) D_q is a monotonically increasing function of $q/(q-1)$.

(*b*) $D_q = \alpha(q) = f(q) = 1$ at $q=1$ implies $m \to 0$ as $\log[q/(q-1)] \to \infty$.

(*c*) The limiting behaviour $f_{q\to\infty} \to 0$ and $D_{q\to\infty} \to \alpha_{\min} \neq 0$ implies $m \to 1$ as $\log[q/(q-1)] \to 0$.

(*d*) Finally, the approximation $D_q \sim \alpha_q$, which is true even for relatively small values of q ($q \sim 6$), together with the conditions $\mathrm{d}\alpha/\mathrm{d}q < 0$, $\mathrm{d}f/\mathrm{d}\alpha = q$ and $D_q \geqslant 0$ yield the result $\mathrm{d}m/\mathrm{d}\{\log[q/(q-1)]\} < 0$.

This monotonic and limiting behaviour of D_q and m disallows any possible reconciliation between D_q and $D\gamma_j$, in precisely the limit we should be expecting it, for large q.

Using the above, we extrapolate D_q (curve 1, figure 3) to obtain the estimate $\alpha_{\min}(q \to \infty) \sim 0.65$. This predicts the asymptotic value for the fractal dimension of DLA to be $D = 1 + \alpha_{\min} \sim 1.65$ [19, 20], quite different from the numerical estimates of ~ 1.71. In addition, $D = 1 + \alpha_{\min}$ can also be estimated using $\gamma_{j\to\infty}$, obtained numerically by Meakin *et al* [18] for DLA clusters. Using $\gamma_\infty = 1 - 1/D$, with $\gamma_\infty = 0.396$, we get the result $D \sim 1.652$. Hence, we are led to consider the possibility that the asymptotic value for the fractal dimension of off-lattice DLA may in fact be ~ 1.65 instead of ~ 1.71. Assuming, this, we re-evaluate $D\gamma_j$ and present the results in column 4 of table 2, also expressing the data in curve 3, figure 3. In this case the agreement between D_q and $D\gamma_j$ for $q > 5$ is remarkable. The discrepancy for small q is probably due to inaccuracies

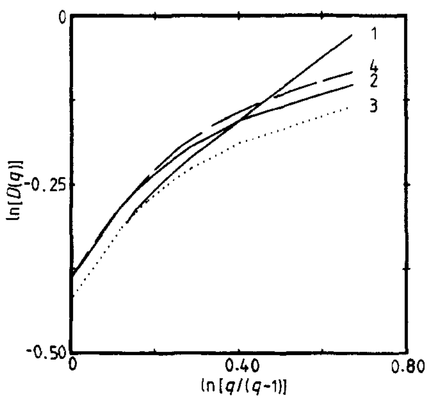

Figure 3. D_q and $D\gamma_j$ for DLA simulations and the deterministic model.

in estimating D_q since these data are less consistent with the expected behaviour of $\ln(D_q) = m = 0$ in the limit $q/(q-1) \to \infty$.

In some sense this extremely slow approach of D to its asymptotic value, hitherto unnoticed, is not surprising in view of the fact that estimates of D for on-lattice DLA decrease very slowly to ~ 1.63 for cluster sizes in excess of 4 million particles [21].

We now compare the harmonic measure for DLA clusters to that of the object in figure 1(a) obtained deterministically. D_q for the latter is given in column 5 of table 2, where the errors represent a 95% confidence limit, and expressed in curve 4, figure 3. Examining table 2, we note that the agreement with $D\gamma_j$ for $D = 1.71$ is remarkable, though both results are transients. Though we cannot say conclusively that $\alpha_{\min} \to 0.65$ for the deterministic model, the estimated bounds $0.627 < \alpha_{\min} < 0.684$ seem encouraging, in which case we have very good agreement between D_q for the deterministic model and both D_q and $D\gamma_j$ (with $D = 1.65$) for DLA clusters. Hence, the similarities between DLA clusters and the deterministic model go beyond the fractal and ramified structure, to the extent of the metric properties of the probability measures.

In conclusion, we emphasise the main results in this letter. In contradiction to available estimates from numerical simulations, our results suggest the asymptotic value of the fractal dimension for off-lattice DLA to be $D \sim 1.65 \pm 0.03$. However, we emphasise that this result is not conclusive since there is a substantive noteworthy literature claiming this value to be ~ 1.71. Clearly, a more detailed study of the fractal dimension and the multi-fractal properties of DLA is needed to resolve this issue, especially since the difference between the two estimates is not substantial and estimating the multi-fractal spectrum of DLA is a very difficult problem. In addition, we estimate the limiting value for the fractal dimension of the object generated by the deterministic growth process to be in the range $1.672 < D < 1.684$. Finally, we show that the similarities between the deterministic model and simulated DLA clusters extend beyond their fractal dimensions to the extent of the metric properties of the growth probability distribution on their interfaces. This gives us a better insight into the role of noise in DLA. Whether this is sufficient to conclude that extrinsic noise is irrelevant in DLA is not immediately clear and can be addressed only to the extent to which the multifractal characterisation of the growth process is unique.

This work was supported by DoE grant no DEFG02-85ER54189.

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